Math 371
Spring 2019
Practice Midterm 2
4/3/2019
Time Limit: 80 Minutes

Name: ____________________

ID ____________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ____________________

This exam contains 10 pages (including this cover page) and 9 questions.
Total of points is 108.

- Check your exam to make sure all 10 pages are present.
- You may use writing implements and a single handwritten sheet of 8.5"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
6	12	
7	12	
8	12	
9	12	
Total:	108	

1. (12 points) State the definition of an ideal of a ring. Find all the ideals in $\mathbb{Z}/6\mathbb{Z}$.

I is an ideal of R iff

① I is an additive subgroup

② $\forall r \in R$, $a \in I$, $a \cdot r \in I$

$\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$.

$\{\text{ideals of } \mathbb{Z}/6\mathbb{Z}\} \leftrightarrow \{\text{ideals of } \mathbb{Z} \text{ containing } 6\mathbb{Z}\}$.

(a) contains (6) iff $6 = a \cdot b$.

So all the ideals are

$(\bar{0})$, $(\bar{2})$, $(\bar{3})$, $\mathbb{Z}/6\mathbb{Z}$

(or written as (0), (2), (3), (1))

2. (12 points) Find the units in $\mathbb{Z}/9\mathbb{Z}$.

an element $x \in \mathbb{Z}/9\mathbb{Z}$ is a unit iff x is not a zero divisor.

which means $(x, 9) = 1$.

So all the units are

1, 2, 4, 5, 7, 8.

3. (12 points) Is $(i+4)$ a maximal ideal in $\mathbb{Z}[i]$? Why?

$$\mathbb{Z}[i]/(i+4) = \mathbb{Z}[x]/x^2+1 \Big/ (x+4)$$

$$= \mathbb{Z}[x]/(x+4,\ x^2+1)$$

$t = x+4$

$$= \mathbb{Z}[t]/(t,\ (t-4)^2+1)$$

$$= \mathbb{Z}[t]/(t,\ t^2-4t+17)$$

$$= \mathbb{Z}[t]/(t,\ 17)$$

$$= \mathbb{Z}/17\mathbb{Z}$$

17 is a prime number.

So $\mathbb{Z}/17\mathbb{Z}$ is a field.

So $(i+4)$ is a maximal ideal

4. (12 points) What are the maximal ideals of $\mathbb{C}[x,y]/(xy,(x-2)(y-1))$?

Hilbert's Nullstellensatz

$\Rightarrow$ maximal ideals of

$\mathbb{C}[x,y]/(xy,(x-2)(y-1))$ corresponds

to $\begin{cases} xy=0 \\ (x-2)(y-1)=0 \end{cases}$

$\begin{cases} x=0 \text{ or } y=0 \\ x=2 \text{ or } y=1. \end{cases}$

So $x=0,\ y=1$ or $x=2,\ y=0$

Maximal ideals are $(x, y-1)$

$(x-2, y)$.

5. (12 points) Find the kernel of the homomorphism $\mathbb{C}[x,y] \to \mathbb{C}[t]$ determined by $x \mapsto t^2+t, y \mapsto t-1$.

Use change of variables.

$X = x$. $Y = y+1$.

then $\varphi(X) = t^2+t$. $\varphi(Y) = t$.

$\varphi(X - Y^2 - Y) = 0$.

Claim $\ker\varphi = (X - Y^2 - Y) = (x-(y+1)^2-(y+1))$
$= (x - y^2 - 3y - 2)$

If $f(X,Y) \in \ker\varphi$.

$f(X,Y) = q(X,Y)\cdot(X - Y^2 - Y)$
$+ r(X,Y)$

$\deg_X r(X,Y) < \deg_X (X - Y^2 - Y) = 1$.

So $r(X,Y) = r(Y)$

$\varphi(f) = 0 \Rightarrow \varphi(r(X,Y)) = 0 \Rightarrow r(t) = 0$

$\Rightarrow r(X,Y) = 0$. so $f(X,Y) \in (X - Y^2 - Y)$

6. (12 points) Give an example of irreducible polynomial $f(x)$ of degree 2 in $\mathbb{F}_3[x]$. Use $f(x)$ to construct an example of a field consisting of 9 elements.

$f(x) = x^2 + ax + b.$

$f(x)$ is irreducible iff $f(x)$ does not have a divisor of degree 1.

So $f(x) \neq (x-m)(x-n)$

In other words $f(m) \neq 0$ for any $m \in \mathbb{F}_3$.

choose $f(x) = x^2 + a$.

$m = 0 \quad 1 \quad 2$

$m^2 = 0, 1, 1.$

so we can choose $a = 1$.

$f(x) = x^2 + 1$. is irreducible

$\mathbb{F}_3(x) / (f(x))$ is a field with 9 elements

(Because $f(x)$ is irreducible, $\mathbb{F}_3[x]$ is PID, so $(f(x))$ is a maximal ideal)

7. (12 points) State the definition of prime element in an integral domain R. Find all the prime elements in $\mathbb{C}[t]$

Defn: If p divides ab, then p divides a or p divides b.

(or, $R/(p)$ is an integral domain)

$\mathbb{C}[t]$ is PID. so any prime element is also an irreducible element.

$f(t)$ is irreducible if and only if $\deg f(t) = 1$.

so $f(t) = ax+b$. $a \neq 0$.

8. (12 points) Prove that $\mathbb{Z}[i]/(3)$ is a field.

$$\mathbb{Z}[i]/(3) = \mathbb{Z}[x]/(x^2+1, 3)$$

$$= \mathbb{Z}[x]/(3) \Big/ (x^2+1)$$

$$= \mathbb{F}_3[x]/(x^2+1)$$

Since $0^2+1 \neq 0$. $1^2+1 = 2$, $2^2+1 = 1$.

So $f(x) = x^2+1$ has no degree-one divisor in $\mathbb{F}_3[x]$.

So $f(x)$ is irreducible

This implies that (x^2+1) is a maximal ideal in $\mathbb{F}_3[x]$, so $\mathbb{Z}[i]/(i+3)$ is a field.

9. (12 points) Let $f = x^3 + x^2 + x + 1$ and let α denote the residue of x in the ring $R = \mathbb{Z}[x]/(f)$. Express $(\alpha^4 + \alpha)(\alpha + 1)$ in terms of the basis $(1, \alpha, \alpha^2)$ of R.

$$(\alpha^4 + \alpha)(\alpha + 1)$$

$$= \alpha^5 + \alpha^4 + \alpha^2 + \alpha$$

$$= \alpha^2(\alpha^3 + \alpha^2 + \alpha + 1) - \alpha^2 \cdot \alpha^2 - \alpha^2 \cdot \alpha - \alpha^2 \cdot 1$$
$$+ \alpha^4 + \alpha^2 + \alpha$$

$$= -\alpha^3 + \alpha$$

$$= -(\alpha^3 + \alpha^2 + \alpha + 1) + \alpha^2 + \alpha + 1$$
$$+ \alpha$$

$$= \alpha^2 + 2\alpha + 1.$$